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Extremals of curvature energy actions on spherical closed curves

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Dedicated to Prof. B.-Y. Chen on occasion of his 60th birthday

Abstract

We compute the first integrals of the Euler–Lagrange equations associated to actions of curvature energy functionals on suitable spaces of curves in real space forms. Then we obtain closure conditions for critical points fully immersed in the three-sphere and apply the results to specific functionals of geometric interest.

MSC: 53A04; 53B20; 49J05; 74K10

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1. Introduction

We consider the variational problem associated to functionals of the form $\mathbf{\Theta}(\gamma) = \int_\gamma P(\kappa)\,\mathrm{d}s$ (*curvature energy functionals*) when acting on suitable spaces of curves in a Riemannian manifold. Here κ denotes the curvature of a curve γ and $P(\kappa)$ is a smooth function. The geometric importance of minimizing functionals of this type defined on a space of curves in the three-dimensional Euclidean space $\mathbb{R}^3$, was pointed out by Blaschke in his book on Differential Geometry [8], where it is referred to as Radon's problem. A more modern reference can be found in the book [16], where the problem is used as a strong motivating example within a more general framework. Apart from its own intrinsic interest, this kind of functionals have a considerable number of applications ranging from

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the construction of mathematical models in Physics to the study of higher dimensional variational problems in submanifolds. Thus, to cite just a few examples, one has that the dynamics of a class of relativistic particles may be described on the basis of actions defined by $P(\kappa) = a\kappa + b$, $a, b \in \mathbb{R}$ (see [14,25] and the references therein). Also, a classical example is provided by $P(\kappa) = \kappa^2$. In 1742, D. Bernoulli posed the extrema of $\int_\gamma \kappa^2\,ds$ as a model for plane elastic curves, which were later classified by L. Euler in his 1744 book on the Calculus of Variations. Much more recently, the same *elastica* model has been extensively examined in the context of a general Riemannian manifold under different points of view [10,17,19–21,24]. To give just one more example, we mention here the critical points of energies given by $P(\kappa) = \kappa^r$ (usually called (free) *r-elastic curves or hyperelastic curves*) in spheres. They have been employed in connection with the Chen–Willmore functional [11,12] to furnish reduction methods in constructing Chen–Willmore submanifolds [2,5,7,26], and to study related conformal string theories [6]. Effectiveness of these methods is based on the ability to explicitly determine hyperelastic curves in spheres. While examples of closed hyperelastic curves in $\mathbb{S}^3$ with constant curvatures were given in [5], we will obtain in Sections 5 and 6 closed hyperelastic curves with non-constant curvature.

In Section 2 we give the Euler–Lagrange equations corresponding to the actions of $\mathbf{\Theta}(\gamma) = \int_\gamma P(\kappa)\,ds$ on curves in real space forms. They are not really new since several special cases had been previously computed by using different approaches (see remark after Proposition 2). Expressing them in terms of the Riemannian invariants of the curve, one can easily conclude that the dimension of the ambient space must be at most three. Inspired in the Langer and Singer's work on the Kirchhoff elastic rod [23], in this section we also obtain by geometrical means the first integrals of the Euler–Lagrange equations of curvature energy functionals. Ideally, one would like to explicitly integrate them if possible in order to completely determine the critical points. The second variation formula has been computed in [1]. It is a complicate expression which can be simplified in some particular cases [4].

From the geometric point of view one of the most interesting problems is the existence of closed critical points. In this paper, one of our main concerns will be the study of the closed critical curves which are fully immersed in the three-sphere. Thus in Section 3 we shall establish conditions under which curves in $\mathbb{S}^3$ corresponding to periodic solutions of the Euler–Lagrange equations close up and then, the subsequent sections are devoted to apply the results previously obtained to concrete examples of special geometric significance. Curves in the two-sphere were considered in [4]. The work we develop here is a natural continuation of it and will show that there are significative differences between the two- and three-dimensional cases.

In Section 5 we consider *curvature energy* functionals of the type $\mathcal{F}^r(\gamma) = \int_\gamma \kappa^P r\,ds$ acting on curves of the three-dimensional sphere $\mathbb{S}^3$. This seems to be a natural choice since there are no closed (free) hyperelastic curves in $\mathbb{R}^3$ [1]. If $r = 0$, critical points are simply geodesics. If $r = 1$, $\mathcal{F}(\gamma) = \int_\gamma \kappa$ is the *total curvature* functional and its only closed critical points are the horizontal lifts, via the Hopf map, of m-covered closed curves in $\mathbb{S}^2(1/2)$ whose enclosed area satisfies a suitable condition [2]. As a consequence, there are no critical points in the two-dimensional sphere. For $r = 2$, $\mathcal{F}^2(\gamma) = \int_\gamma \kappa^2\,ds$ is nothing but the classical Bernoulli's *elastica* functional which has been extensively studied in the literature as we mentioned before. In this section we completely determine the space of closed *elasticae* in $\mathbb{S}^3$. It can be parameterized (modulo isometries and multiple covers of the curves) in the plane region $\bar{\mathbf{A}} \cap \mathbb{Q}^2$, where $\mathbf{A} = \{(x, y); x^2 + y^2 < 1/2, 0 < x$ and $y < -(1/2)\}$ (see Fig. 5). Points in the upper border of this region correspond to closed (free) elastic curves in $\mathbb{S}^2$ (closed elastic curves in $\mathbb{S}^2$ had been classified in [20]). Points in the lower border of $\bar{\mathbf{A}} \cap \mathbb{Q}^2$ correspond to closed elastic helices fully immersed in $\mathbb{S}^3$ (we use the term *helices* for curves with non-null constant Frenet curvatures) which are totally describe in Section 5.1 [5]. Points inside the region $\mathbf{A} \cap \mathbb{Q}^2$ correspond to closed elastic curves fully immersed in $\mathbb{S}^3$ with non-constant curvature. If r is any natural number greater than two, a explicit construction of closed examples is much more difficult. By using a similar method to that of previous cases, one can also determine in this case the closed hyperelastic helices. However, existence of closed non-constant curvature hyperelastic curves in $\mathbb{S}^3(1)$ should not be taken for granted. For instance, there are no closed hyperelastic curves in $\mathbb{S}^2$ for $r > 2$ [4]. Although we can prove in Section 6 that for any natural number $r > 2$ there exist closed critical hyperelastic curves in $\mathbb{S}^3$ with non-constant curvature, a general explicit formula for their curvatures seems to be elusive for the moment (with the exception of some particular low values of r, for which some reductions in the ODE degrees may be possible).

For another application of the general computations, we consider in Section 4 the family of functionals $\mathbf{\Theta}^\lambda(\gamma) = \int_\gamma (\kappa^2 + \lambda)^{1/2}$, $0 \le \lambda$ acting on curves γ of $\mathbb{S}^3(G)$, the three-sphere of curvature G. When $\lambda = G$ this functional is nothing but that the *total curvature* of γ in $\mathbb{R}^4$ constrained to act on spherical curves. This type of functionals have also appeared in the study of a model of relativistic particle with maximal proper acceleration [25]. From the perspective of the existence of closed critical curves, the most interesting case is when $\lambda \in (0, G)$. Being this the case, we completely determine in Section 4.2, the space of closed curves in $\mathbb{S}^3(1)$ which are critical points of $\mathbf{\Theta}^\lambda$. This time, it can be parameterized (modulo isometries and multiple covers of the curves) in the plane region $\bar{\mathbf{A}} \cap \mathbb{Q}^2$, where now $\mathbf{A} = \{(x, y); x < 0$ and $-(1/4x\sqrt{1-\lambda}) < y < 1/2\}$ (see Fig. 4). Points in the upper border of this region correspond to closed critical curves in $\mathbb{S}^2(1)$ and have been studied in [4]. Points in the lower border of $\bar{\mathbf{A}} \cap \mathbb{Q}^2$ correspond to closed critical helices fully immersed in $\mathbb{S}^3(1)$ which we describe in Section 4.2, while points inside the region $\mathbf{A} \cap \mathbb{Q}^2$ correspond to non-constant curvature closed critical curves fully immersed in $\mathbb{S}^3(1)$.

2. Euler–Lagrange equations in space forms

We first consider general *curvature energy functionals* acting on spaces of curves satisfying given boundary conditions in an n-dimensional Riemannian manifold $\mathbf{M}^n$ with metric $\langle\, ,\, \rangle$, Levi–Civita connection ∇, and curvature tensor R. We shall denote by $\mathbf{\Omega}_{pq}$ the space of curves in $\mathbf{M}^n$, satisfying the following conditions: (1) $\gamma : \mathbf{I} = [0, 1] \to \mathbf{M}^n$, $\gamma \in C^4(\mathbf{I})$; (2) γ is a immersed curve in $\mathbf{M}^n$, that is, $\mathrm{d}\gamma/\mathrm{d}t \neq 0$; (3) they all have the same ends, $\gamma(0) = p$, $\gamma(1) = q$, $\forall \gamma \in \mathbf{\Omega}_{pq}$, where $p, q \in \mathbf{M}^n$; (4) there is a well defined normal vector on γ (for instance, $\mathrm{d}^2\gamma/\mathrm{d}t^2 \neq 0$ if $n > 2$, or $n = 2$ and $\mathbf{M}^2$ is orientable). We take $P(t)$ a C^∞ function and consider the following *curvature energy functional* $\mathbf{\Theta}(\gamma) = \int_\gamma P(\kappa)$ acting on $\mathbf{\Omega}_{pq}$. For a given curve γ in $\mathbf{\Omega}_{pq}$, we denote by $T(t)$, $N(t)$ the unit tangent and normal vectors, respectively. Also $\kappa^2(t) = \|\nabla_T T\|^2$ will be the squared curvature and κ will be the oriented curvature if γ is a curve in an oriented surface $\mathbf{M}^2$ and the positive root of κ^2 otherwise. As usual the arclength parameter is represented by $s \in [0, L]$, L being the length of γ. Let us take a curve $\Gamma(w)$ in $\mathbf{\Omega}_{pq}$ passing through γ, that is $\Gamma(w, t) = \gamma_w(t) : (-\varepsilon, \varepsilon) \times \mathbf{I} \to \mathbf{M}^n$ is a variation of γ in $\mathbf{\Omega}_{pq}$ with $\gamma(0, t) = \gamma(t)$, whose variation vector field along the curve γ is given by $W = W(t) = (\partial\gamma/\partial w)(0, t)$. The restriction of a *curvature energy functional* to a variation is denoted by the same letter, $\mathbf{\Theta}(w) = \mathbf{\Theta}(\gamma_w(t))$. To compute the first derivative of $\mathbf{\Theta}(w)$ we shall use the following notation:

$$\mathcal{K} = P'(\kappa) \cdot N, \qquad \mathcal{J} = \nabla_T \mathcal{K} + \big(2\kappa P'(\kappa) - P(\kappa)\big) \cdot T,$$
$$\mathcal{E} = \nabla_T \mathcal{J} + P'(\kappa) \cdot R(N, T)T, \tag{1}$$

where $P'(\kappa) = \mathrm{d}P/\mathrm{d}\kappa$. Then using Lemma 1 and Proposition 2.1 of [20], the first *Frenet formula*, $\nabla_T T = \kappa N$, and integrating by parts one can obtain [1].

Proposition 1. *Let $\Gamma(w, s) = \gamma_w(t)$ be a variation of a γ by curves in $\mathbf{\Omega}_{pq}$. Under the above conditions and notation, the following formula holds*:

$$\frac{\mathrm{d}}{\mathrm{d}w}\mathbf{\Theta}(w)_{|w=0} = \left(\int_0^L \langle \mathcal{E}, W\rangle \,\mathrm{d}s\right) + \mathcal{B}[W, \gamma]_0^L, \tag{2}$$

where the boundary term is given by

$$\mathcal{B}[W, \gamma]_0^L = [\langle \mathcal{K}, \nabla_T W\rangle - \langle \mathcal{J}, W\rangle]_0^L. \tag{3}$$

Now, we may see $\mathbf{\Theta}$ acting on spaces of curves which, in addition to the defining conditions of $\mathbf{\Omega}_{pq}$, satisfy also a suitable set of boundary conditions. We first consider $\tilde{\mathbf{\Omega}}_{pq}$, the subspace of curves in $\mathbf{\Omega}_{pq}$ which also verify: $(\mathrm{d}\gamma/\mathrm{d}s)(0) = e_1$, $(\mathrm{d}\gamma/\mathrm{d}s)(1) = e_2$, where e_1 and e_2 are two fixed tangent vectors to $\mathbf{M}^n$ at p and q, respectively. The following computations might be equally applied, for instance, to the case in which $\mathbf{\Theta}$ acts on either $\mathbf{\Omega}$, the space of smooth closed curves of $\mathbf{M}^n$, or to $\hat{\mathbf{\Omega}}_{pq}$, the space of curves with "clamped" ends (that is, with fixed *Frenet frame* at each end). In such cases, the above boundary term drops out. Thus a critical point of $\mathbf{\Theta}$ in such spaces will be characterized by the *Euler–Lagrange equation* $\mathcal{E} = 0$, in other words by

$$\nabla_T^2(P'(\kappa) \cdot N) + \nabla_T((2\kappa P'(\kappa) - P(\kappa)) \cdot T) + P'(\kappa) \cdot R(N, T)T = 0. \tag{4}$$

From now on, we restrict ourselves to the case in which our manifold $\mathbf{M}^n$ is a simply connected real space form of constant curvature G: $\mathbf{M}^n(G)$. The symmetry of these spaces will simplify the *Euler–Lagrange equation* to the point where some reductions are possible. We write down the *Frenet equations* as: $\nabla_T T = \kappa \cdot N$, $\nabla_T N = -\kappa \cdot T + \tau \cdot B$, $\nabla_T B = -\tau \cdot N + \eta$, where τ stands for the second Frenet curvature (*torsion*) and η is perpendicular to the bundle spanned by $\{T, N, B\}$. Let us denote by $\mathcal{H}$ any of the following spaces of curves in $\mathbf{M}^n(G)$: $\tilde{\mathbf{\Omega}}_{pq}$, $\mathbf{\Omega}$ or $\hat{\mathbf{\Omega}}_{pq}$. Then, by substituting the *Frenet equations* into the Euler–Lagrange equation (4), comparing the components of $\{T, N, B, \eta\}$ and making use of Erbacher's theorem on the reduction of codimension in a real space form [13], one gets the following proposition.

Proposition 2. *Let $\mathbf{\Theta}$ be a curvature energy functional acting on $\mathcal{H}$. Then γ is a critical point of $\mathbf{\Theta}$, if and only if*

$$(\kappa^2 - \tau^2 + G)P'(\kappa) + \frac{\mathrm{d}^2 P'(\kappa)}{\mathrm{d}s^2} = \kappa P(\kappa), \tag{5}$$

$$2\frac{\mathrm{d}P'}{\mathrm{d}s}\tau + P'(\kappa)\tau_s = 0, \tag{6}$$

$$P'(\kappa) \cdot \eta = 0, \tag{7}$$

where κ_s, τ_s denote derivative with respect to the arclength parameter s. Moreover γ must lie fully in either a two-dimensional or a three-dimensional totally geodesic submanifold of $\mathbf{M}^n(G)$.

Several versions of the above Euler–Lagrange equations are known in the literature. For instance, they were found for curves in $\mathbb{R}^3$ in [18]; for curves in Lorentzian ambient spaces in [25]; and for curves in Riemannian space forms in [1] (see also [16]).

From the above Proposition, we may assume that γ is a curve in $\mathbf{M}^n(G)$ with $n = 2, 3$. If $P''(\kappa) = 0$, then this case basically corresponds either to the length or to the total curvature functionals and it has been studied in [1,3]. Assume $P''(\kappa) \neq 0$. The next proposition gives a frame adapted to our problem, which will allow us to use the Noether argument relating symmetries of $\mathbf{\Theta}$ to constants of motion along γ to obtain the first integrals of the Euler–Lagrange equations [1].

Let $\gamma : [0, 1] \to \mathbf{M}^3(G)$ be a immersed regular curve in a three-dimensional real space form of constant sectional curvature G. A vector field W is called *Killing* along γ, if for any variation in the direction of W we have

$$\frac{\partial v}{\partial w} = \frac{\partial \kappa}{\partial w} = \frac{\partial \tau}{\partial w} = 0. \tag{8}$$

Now assume $\kappa_s \neq 0$. By using Lemma 1 of [20], one can check that W is a Killing field along γ, if and only if

$$\langle \nabla_T W, T\rangle = 0, \tag{9}$$

$$\langle \nabla_T^2 W, N\rangle + G\langle W, N\rangle = 0, \tag{10}$$

$$\left\langle \frac{1}{\kappa}\nabla_T^3 W - \frac{\kappa_s}{\kappa^2}\nabla_T^2 W + \left(\frac{G}{\kappa} + \kappa\right)\nabla_T W - \frac{\kappa_s}{\kappa^2} G W, B\right\rangle = 0. \tag{11}$$

Moreover, it was proved in [20] that a Killing field along a curve γ in a real space form $\mathbf{M}^3(G)$ is the restriction to γ of a Killing field on $\mathbf{M}^3(G)$. Define a new vector field $\mathcal{I} = \mathcal{K} \times T$ on γ, then from (1) we have

$$\mathcal{J} \times T = \nabla_T(\mathcal{K} \times T), \tag{12}$$

$$\mathcal{I} = -P'(\kappa) \cdot B. \tag{13}$$

Thus, one gets the following proposition.

Proposition 3. *Let $\gamma : [0, 1] \to \mathbf{M}^3(G)$ be a immersed curve in a real space form of constant sectional curvature G which is a critical point of $\boldsymbol{\Theta}$ (under any boundary conditions). Assume that $(\mathrm{d}P'(\kappa)/\mathrm{d}s) \neq 0$ along γ, then the vector fields $\mathcal{J}$ and $\mathcal{I}$, defined in (1) and (13), respectively, are Killing fields along γ (and therefore, they are the restriction to γ of Killing fields on $\mathbf{M}^3(G)$).*

Proof. From the definition of $\mathcal{I}$ and the Frenet equations, the tangent part of $\nabla_T\mathcal{I}$ is obviously zero, which gives (9). The component in N of $\nabla_T^2\mathcal{I}$ is $(\mathrm{d}P'/\mathrm{d}s)\tau + (\mathrm{d}(P'\tau)/\mathrm{d}s)$, which is zero because of (6). This fact and (13) gives (10). Finally taking $W = \mathcal{I}$, the term on the left of (11) is $(\mathrm{d}/\mathrm{d}s)[(1/\kappa)(P'\tau^2 - (\mathrm{d}^2P'/\mathrm{d}s^2) - GP')] - \kappa(\mathrm{d}P'/\mathrm{d}s)$, which is also zero as a consequence of (5). This gives (11) and shows that $\mathcal{I}$ is a Killing field on γ. On the other hand, from the second and third equations of (1), we see that since γ is a critical point

$$\mathcal{J} = (\kappa P'(\kappa) - P(\kappa)) \cdot T + \frac{\mathrm{d}P'(\kappa)}{\mathrm{d}s} \cdot N + \tau P'(\kappa) \cdot B, \tag{14}$$

$$\nabla_T\mathcal{J} = -GP'(\kappa) \cdot N, \tag{15}$$

which immediately gives (9). Now, the component in N of $\nabla_T^2\mathcal{J}$ is $-G(\mathrm{d}P'/\mathrm{d}s)$, which in combination with (14) shows (10). Finally, we see from (15) that the component in B of $\nabla_T\mathcal{J}$ is zero. By using the Frenet equations and (6) one can see also that the binormal component of $\nabla_T^3\mathcal{J}$ is also zero. Moreover the binormal component of $\nabla_T^2\mathcal{J}$ is $-G\tau P'$, which together with (14) gives (11). Therefore, $\mathcal{J}$ is also a Killing field on γ. □

Now, if γ happens to be a critical point of $\boldsymbol{\Theta}$ (under any boundary conditions), then standard arguments imply that $\mathcal{E} = 0$ on γ. The variation formula continue to hold when L is replaced by any intermediate value $t \in (0, L)$ and, thus, the first variation formula (2) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}w}\boldsymbol{\Theta}(\gamma)_{|w=0} = \mathcal{B}[W, \gamma]_0^t. \tag{16}$$

Therefore, for any Killing field W on $\mathbf{M^3}(G)$, we have $\mathcal{B}[W, \gamma]_0^t = 0, \forall t \in [0, L]$. Thus we see from (3) that $\langle\mathcal{K}, \nabla_T W\rangle - \langle\mathcal{J}, W\rangle$ is constant on γ. Then using Proposition 3, we have

$$\langle\mathcal{K}, \nabla_T\mathcal{I}\rangle - \langle\mathcal{J}, \mathcal{I}\rangle = c, \tag{17}$$

on γ, where $c \in \mathbb{R}$ is a constant. Combining (1) and $\langle\mathcal{I}, \mathcal{K}\rangle = 0$, we have

$$\langle\nabla_T\mathcal{I}, \mathcal{K}\rangle + \langle\mathcal{I}, \mathcal{J}\rangle = 0. \tag{18}$$

Finally, combining also (17) and (18), one obtains

$$\langle\mathcal{I}, \mathcal{J}\rangle = -e, \tag{19}$$

on γ, where $e \in \mathbb{R}$ is a constant. By a similar argument, one can see that

$$\langle\mathcal{J}, \mathcal{J}\rangle + G\langle\mathcal{I}, \mathcal{I}\rangle = d, \tag{20}$$

on γ, where d is a constant. Now, plug (1) in (19) and (20) to obtain the following proposition.

Proposition 4. *Let $\gamma : [0, 1] \to \mathbf{M}^3(G)$ be a non-constant curvature regular curve immersed in a real space form of constant sectional curvature G. Assume that $P''(\kappa) \neq 0$ and that γ is a critical point of $\mathbf{\Theta}$ in $\mathcal{H}$ with non-constant curvature. Then, with the above notation, we have*

$$e = \tau(P'(\kappa))^2, \tag{21}$$

$$d = (P''(\kappa))^2\kappa_s^2 + (\kappa P'(\kappa) - P(\kappa))^2 + G(P'(\kappa))^2 + \frac{e^2}{(P'(\kappa))^2}. \tag{22}$$

Observe that the first of these equations can be obtained directly by multiplying (6) by $P'(\kappa)$. Integration of the Frenet equations can be an enormous task even for simple choices of $P(\kappa)$ in $\mathbb{R}^3$. To simplify the problem we shall need the following result.

Proposition 5. *Let $\gamma : [0, 1] \to \mathbf{M}^3(G)$ be an isometrically immersed curve in a real space form of constant sectional curvature G which is a critical point of $\mathbf{\Theta}$ acting on $\mathcal{H}$. Then the extension to $\mathbf{M}^3(G)$ of Killing fields along γ, $\mathcal{J}$ and $\mathcal{I}$ (defined in (1) and (13), respectively) commute, that is, $[\mathcal{I}, \mathcal{J}] = 0$.*

Proof. Let us denote by $\{T, N, B\}$ the Frenet frame along $\gamma(s)$ and take of a variation of γ with variation field $\mathcal{I}$. Since $\mathcal{I}$ is Killing field on $\mathbf{M}^3(G)$, we have

$$\mathcal{I}\langle T, \mathcal{J}\rangle = \langle[\mathcal{I}, \mathcal{J}], T\rangle + \langle[\mathcal{I}, T], \mathcal{J}\rangle. \tag{23}$$

From (14) we have $\langle T, \mathcal{J}\rangle = \kappa P'(\kappa) - P(\kappa)$, then (8) gives

$$\mathcal{I}\langle T, \mathcal{J}\rangle(s) = 0. \tag{24}$$

Moreover, from formula 3 of Lemma 1 in [20] and (13), we have

$$[\mathcal{I}, T](s) = 0. \tag{25}$$

Therefore, combining (23)–(25), we obtain

$$\langle[\mathcal{I}, \mathcal{J}], T\rangle(s) = 0. \tag{26}$$

Using again that $\mathcal{I}$ is Killing field and (13), (14) and (8), we get

$$0 = \mathcal{I}\langle \mathcal{I}, \mathcal{J}\rangle(s) = \langle[\mathcal{I}, \mathcal{J}], \mathcal{I}\rangle(s) = -P'(\kappa)\langle[\mathcal{I}, \mathcal{J}], B\rangle(s). \tag{27}$$

On the other hand, since $[\mathcal{I}, \mathcal{J}]$ is also a Killing field $\langle\nabla_T[\mathcal{I}, \mathcal{J}], T\rangle = 0$, which along with (26) implies

$$\langle[\mathcal{I}, \mathcal{J}], N\rangle(s) = 0. \tag{28}$$

We see from (26)–(28) that $[\mathcal{I}, \mathcal{J}](s) = 0$ on $\gamma(s)$. If $[\mathcal{I}, \mathcal{J}]$ were not identically zero on $\mathbf{M}^3(G)$, then $\gamma(s)$ would be included in a connected component of the set of fixed points of the one-parameter group associated to $[\mathcal{I}, \mathcal{J}]$. This space is a totally geodesic submanifold of $\mathbf{M}^3(G)$ and therefore, $\gamma(s)$ would not be fully immersed in $\mathbf{M}^3(G)$ which is a contradiction. □

3. Closed critical points in three-spheres

From now on, we assume that $\mathbf{M}^3(G) = \mathbb{S}^3(G)$ and that $P''(\kappa) \neq 0$. For simplicity, we take without loss of generality $G = 1$ in this section. Assume that $\kappa(s)$ is a periodic solution of (5) and (6) with period ρ and that $\gamma(s)$ is the corresponding critical point of $\mathbf{\Theta}(\gamma) = \int_\gamma P(\kappa)$ acting on $\tilde{\mathbf{\Omega}}_{pq}$. Then, we are going to see that the Killing fields $\mathcal{J}$ and $\mathcal{I}$ are naturally related to a system of cylindrical coordinates in the three-sphere where we can express the closure conditions. Assuming $\theta, \varphi \in (0, 2\pi)$, $\psi \in (0, \pi/2)$, we take

$$X(\theta, \varphi, \psi) = (\cos\theta\cos\psi,\ \sin\theta\cos\psi,\ \cos\varphi\sin\psi,\ \sin\varphi\sin\psi), \tag{29}$$

then the vector fields X_θ and X_φ are also Killing vector field that obviously commute. By using the above proposition and after a suitable rotation in $\mathbb{R}^4$, we may assume that

$$\mathcal{I} = aX_\theta + bX_\varphi, \qquad \mathcal{J} = \tilde{a}X_\theta + \tilde{b}X_\varphi, \tag{30}$$

where a and b are the real numbers. Since $\mathcal{I}$ and $\mathcal{J}$ are linearly independent and since formulas (19) and (20) imply that $\langle\mathcal{J},\mathcal{J}\rangle + \langle\mathcal{I},\mathcal{I}\rangle$ and $\langle\mathcal{J},\mathcal{I}\rangle$ are constant on $\gamma(s)$ and that $\langle\mathcal{I},\mathcal{I}\rangle$ is not constant on $\gamma(s)$, we conclude that $\tilde{b} = \pm a$, $\tilde{a} = \pm b$. We may assume that $\tilde{a} = b$, $\tilde{b} = a$, and $a^2 > b^2 \geq 0$. Hence, from (19), (20), (29) and (30), we get

$$X_\theta = \frac{1}{a^2 - b^2}(a\mathcal{I} - b\mathcal{J}), \qquad X_\varphi = \frac{1}{a^2 - b^2}(a\mathcal{J} - b\mathcal{I}), \tag{31}$$

and

$$\begin{aligned} &\langle\mathcal{J},\mathcal{I}\rangle = ab = -e, \qquad \langle\mathcal{J},\mathcal{J}\rangle + \langle\mathcal{I},\mathcal{I}\rangle = a^2 + b^2 = d, \\ &\langle\mathcal{I},\mathcal{I}\rangle - \langle\mathcal{J},\mathcal{J}\rangle = (a^2 - b^2)\cos 2\psi. \end{aligned} \tag{32}$$

Thus

$$\begin{aligned} \cos 2\psi &= \frac{1}{a^2 - b^2}(2\langle\mathcal{I},\mathcal{I}\rangle - d) = 2\frac{(P'(\kappa))^2 - b^2}{a^2 - b^2} - 1, \\ \cos^2\psi &= \frac{1}{2} + \frac{1}{2(a^2 - b^2)}(2\langle\mathcal{I},\mathcal{I}\rangle - d) = \frac{(P'(\kappa))^2 - b^2}{a^2 - b^2}, \\ \sin^2\psi &= \frac{1}{2} - \frac{1}{2(a^2 - b^2)}(2\langle\mathcal{I},\mathcal{I}\rangle - d) = \frac{a^2 - (P'(\kappa))^2}{a^2 - b^2}, \end{aligned} \tag{33}$$

where a, b are determined in terms of the integration constants e, d by

$$a^2 = \tfrac{1}{2}(d + \sqrt{d^2 - 4e^2}), \qquad b^2 = \tfrac{1}{2}(d - \sqrt{d^2 - 4e^2}). \tag{34}$$

Now, we express the curve in terms of the local coordinates (29), $\gamma(s) = X(\theta(s), \varphi(s), \psi(s))$ and $\gamma'(s) = T(s) = \theta'(s)X_\theta + \varphi'(s)X_\varphi + \psi'(s)X_\psi$, then using (34), (31), the last two equations of (33) and the expressions of $\mathcal{J}$ and $\mathcal{I}$ in terms of the invariants of the curve (14), (13), one obtains

$$\theta'(s) = \frac{\langle T, X_\theta\rangle}{\|X_\theta\|^2} = \frac{\langle T, X_\theta\rangle}{\cos^2\psi} = \frac{b(\kappa P'(\kappa) - P(\kappa))}{b^2 - (P'(\kappa))^2}, \tag{35}$$

$$\varphi'(s) = \frac{\langle T, X_\varphi \rangle}{\|X_\varphi\|^2} = \frac{\langle T, X_\varphi \rangle}{\sin^2\psi} = \frac{a(\kappa P'(\kappa) - P(\kappa))}{a^2 - (P'(\kappa))^2}. \tag{36}$$

In case we were able to solve (21) and (22) and explicitly obtain κ, then we would obtain the cylindrical coordinates of the critical curve by integrating again (35) and (36). Interested as we are in closed critical points, we need (21) and (22) to have periodic solutions κ. Assuming this and denoting by ρ the period of κ, the first equation of (33) gives us that ψ is also periodic having a factor of ρ as period. Combination of this fact with (35), (36) and the expression of T in cylindrical coordinates implies the following proposition.

Proposition 6. *Assume that $\kappa(s)$ is a non-constant periodic solution (with period ρ) of the Euler–Lagrange equations (5) and (6). Let $\gamma : [0, 1] \to \mathbb{S}^3(1)$ be the corresponding curve in the unit three-sphere (which is a critical point of $\boldsymbol{\Theta}$ acting on $\tilde{\boldsymbol{\Omega}}_{pq}$). Then γ is a closed curve in $\mathbb{S}^3(1)$, if and only if, the progression angles*

$$\Lambda\theta(\gamma) = \int_0^\rho \frac{b(\kappa P'(\kappa) - P(\kappa))}{b^2 - (P'(\kappa))^2}\, \mathrm{d}s, \tag{37}$$

$$\Lambda\varphi(\gamma) = \int_0^\rho \frac{a(\kappa P'(\kappa) - P(\kappa))}{a^2 - (P'(\kappa))^2}\, \mathrm{d}s \tag{38}$$

are rational multiples of 2π (here a and b are given in (34)). Hence, up to multiple covers of the curve, γ shall close up in n periods of its curvature, if and only if, there exist $l, m, n \in \mathbb{Z}$ $(l, m, n) = 1$, such that $n\Lambda\theta(\gamma) = 2\pi l$ and $n\Lambda\varphi(\gamma) = 2\pi m$.

We would like to apply the results obtained so far to concrete choices of the Lagrangian $P(\kappa)$ (with $P''(\kappa) \neq 0$). Although in many cases we do not need to get explicit expressions of the closure conditions in order to show the existence of closed critical points, this will be required if we wish to classify them.

4. Closed critical points of total curvature type energy functionals

Let $\mathbb{S}^3(G)$ be the three-dimensional sphere of constant Gaussian curvature G. We consider the functional

$$\mathcal{F}^{(\lambda)}(\gamma) = \int_\gamma (\kappa^2 + \lambda)^{1/2}\, \mathrm{d}s, \tag{39}$$

where $\kappa(s)$ is the curvature of $\gamma(s)$ and $\lambda \geq 0$, acting on a suitable space of immersed curves of $\mathbb{S}^3(G)$. Actually we assume that $\lambda > 0$ since the case $\lambda = 0$ corresponds to the total curvature in $\mathbb{S}^3(G)$ and was studied in [3]. From (5) and (6) we see that the Euler–Lagrange equations of this functional are

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\frac{\kappa}{(\kappa^2+\lambda)^{1/2}}\right) + \frac{\kappa(\kappa^2 - \tau^2 + G)}{(\kappa^2+\lambda)^{1/2}} - \kappa(\kappa^2+\lambda)^{1/2} = 0, \tag{40}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\kappa^2}{\kappa^2+\lambda}\tau\right) = 0. \tag{41}$$

4.1. Closed critical curves of constant curvature

We first investigate the existence of closed critical points of constant curvature. Assume also that $G = 1$. If κ is a non-zero constant, then the *Euler–Lagrange* equations (40), (41) reduce to

$$\mathcal{E}(\kappa, \tau) = 1 - \lambda - \tau^2 = 0, \tag{42}$$

hence τ is also constant and critical points are helices. Thus, if $\lambda > 1$ we do not have any critical points with non-zero constant curvature κ. If $1 = \lambda$ (λ equals the curvature of the sphere), then $\tau = 0$, and every circle is a critical point for this functional. Assume $0 < \lambda < 1$, then they must be helices satisfying (42). The point is that, helices in $\mathbb{S}^3(1)$ can be considered as geodesics of Hopf Tori [5,7]. More precisely, let us consider a helix in $\mathbb{S}^3(1)$ of known constant curvature and torsion (κ, τ) and denote by α the circle of $\mathbb{S}^2(1/2)$ with curvature

$$\varrho = \frac{\kappa^2 + \tau^2 - 1}{\kappa}, \tag{43}$$

and by $\mathbf{M}_\alpha$ the Hopf torus obtained as the total lifting of α via the Hopf map. Then $\mathbf{M}_\alpha$ is a flat torus determined by the lattice $\Gamma = \text{span}\{(0, 2\pi), (L, 2A)\}$, where L is the length of α and A is the oriented area enclosed by α in $\mathbb{S}^2(1/2)$. Thus our helix may be seen as the geodesic of slope

$$g = \frac{1-\tau}{\kappa} \tag{44}$$

lying on $\mathbf{M}_\alpha$. Then in order the helix to be closed, there must exist a rational number $q \neq 0$, such that

$$g = q\sqrt{\varrho^2 + 4} - \tfrac{1}{2}\varrho. \tag{45}$$

So given a real number ϱ and a rational number q, we determine g in (45) and we consider the closed helix whose curvature and torsion (κ, τ) are obtained by solving (43) and (44). In order to be a critical point of (39), it must satisfy the Euler–Lagrange equation (42). Hence the point is to find a real number ϱ and a rational number q satisfying $\mathcal{E}(\kappa(\varrho, q), \tau(\varrho, q)) = 0$. In our case, this equation gives

$$\sqrt{\varrho^2 + 4}\left(\frac{1}{4}\left(1 - \sqrt{1-\lambda}\right) - q^2\left(1 + \sqrt{1-\lambda}\right)\right) + \varrho q\sqrt{1-\lambda} = 0. \tag{46}$$

Thus, for any $\lambda \in (0, 1)$ we see that the non-empty open subset $\mathcal{U}(\lambda) \subset \mathbb{R}$

$$\mathcal{U}(\lambda) = \left(-\frac{\left(1-\sqrt{1-\lambda}\right)^2}{2\lambda}, -\frac{\left(1-\sqrt{1-\lambda}\right)}{2\sqrt{\lambda}}\right) \cup \left(\frac{\left(1-\sqrt{1-\lambda}\right)}{2\sqrt{\lambda}}, \frac{1}{2}\right) \tag{47}$$

is such that for any rational number q in $\mathcal{U}(\lambda)$, there exists a unique positive solution ϱ of (46) given by

$$\varrho^2 = \frac{4\left((1/4)\left(1-\sqrt{1-\lambda}\right) - q^2(1+\sqrt{1-\lambda})\right)^2}{(1-\lambda)q^2 - \left((1/4)\left(1-\sqrt{1-\lambda}\right) - q^2\left(1+\sqrt{1-\lambda}\right)\right)^2}.$$

Fig. 1. Stereographic projections of the closed helices $\gamma_{1/5}^{(0.1)}$ and $\gamma_{1/9}^{(0.9)}$.

Therefore, one can solve (43)–(45) to get a unique closed helix satisfying $\mathcal{E}(\kappa(\varrho, q), \tau(\varrho, q)) = 0$. We summarize all the information we have just obtained in the next statement, where uniqueness is understood modulo isometries and multiple covers of closed curves.

Proposition 7. *Let $\mathcal{F}^{(\lambda)} : \mathbf{\Omega} \to \mathbb{R}$ be the energy functional defined in (39) acting on $\mathbf{\Omega}$, the space of closed curves in $\mathbb{S}^3(G)$. Then, we have*:

1. *If $\lambda \geq G$, there are no closed critical points of constant curvature*.
2. *If $0 < \lambda < G$, the set of closed critical points with constant curvature* (*and therefore, also with constant non-zero torsion*: *helices*) *forms a rational one-parameter family determined as indicated above by $\{\gamma_q^{(\lambda)} : q \in \mathcal{U}(\lambda) \cap \mathbb{Q}\}$, where $\mathcal{U}(\lambda)$ is given in (47)*.

Note that we are assuming our curves to be fully immersed in $\mathbb{S}^3(G)$, otherwise geodesics would appear as closed critical points in the above two cases and circles would be also critical points if $\lambda = G$ [4]. Fig. 1 shows the stereographic projection from $\mathbb{S}^3(1)$ of closed $\mathcal{F}^{(\lambda)}$-critical helices, numerically obtained for certain values of λ and q.

4.2. Critical curves of non-constant curvature

Assume now that κ is a non-constant function. From the first integrals of the *Euler–Lagrange* Eqs. (21) and (22) with $P(\kappa) = (\kappa^2 + \lambda)^{1/2}$, one has

$$\kappa_s^2(s) = \left(\frac{\kappa^2 + \lambda}{\lambda\kappa}\right)^2 [(\kappa^2 + \lambda)(d\kappa^2 - e^2(\kappa^2 + \lambda)) - \kappa^2(G\kappa^2 + \lambda^2)], \quad (48)$$

$$\tau(s) = e\left(1 + \frac{\lambda}{\kappa^2(s)}\right), \quad (49)$$

where d and e are the constants of integration. Writing $u(s) = \kappa^2(s)$ and

$$Q(x) = (d - e^2 - G)x^2 + (\lambda d - 2\lambda e^2 - \lambda^2)x - e^2\lambda^2. \quad (50)$$

Eq. (48) becomes

$$u_s^2 = \frac{4}{\lambda^2}(u+\lambda)^2 Q(u). \tag{51}$$

To find closed critical curves, we need this equation to have periodic solutions. This happens precisely if (e, d) belongs to the plane region $\mathcal{D}$, determined by the following conditions:

$$0 < e^2 < G - \lambda, \tag{52}$$

$$\lambda + 2e\sqrt{G-\lambda} < d < e^2 + G. \tag{53}$$

From (52) we see that in order to expect closed critical points we must have $0 \leq \lambda < G$.

Proposition 8. *For any $\lambda \geq G$, there exist no non-constant curvature closed critical points of $\mathcal{F}^{(\lambda)}$ in $\mathbb{S}^3(G)$.*

Thus from now on we may assume that λ is a fixed real number satisfying $0 < \lambda < G$ and that the integration constants verify $(e, d) \in \mathcal{D}$. Then $Q(u)$ has two real roots $\alpha > \beta > 0$ and (using for example, formula 2.266 of [15]) we see that

$$u_{d,e}^{(\lambda)}(s) = (\kappa_{d,e}^{(\lambda)})^2(s) = \frac{2\lambda(G-\lambda)}{(2G-d-\lambda) - \sqrt{\vartheta}\sin(2\sqrt{G-\lambda}s - \pi/2)} - \lambda, \tag{54}$$

where $\vartheta = (d-\lambda)^2 - 4e^2(G-\lambda)$, is a two-parameter family of periodic solutions of (51) with initial condition $u_{d,e}^{(\lambda)}(0) = \beta$. Roots of $Q(u)$ give the minima and maxima of the above solutions. They are reached at $u_{d,e}^{(\lambda)}(0) = \beta$ and $u_{d,e}^{(\lambda)}(\pi/2\sqrt{G-\lambda}) = \alpha$, respectively and they can be expressed in terms of the (e, d)-parameters as

$$\alpha = \frac{\lambda(d-2e^2-\lambda) + \lambda\sqrt{\vartheta}}{2(e^2-d+G)}, \qquad \beta = \frac{\lambda(d-2\,e^2-\lambda) - \lambda\sqrt{\vartheta}}{2(e^2-d+G)}. \tag{55}$$

Now, for any periodic function $\kappa_{d,e}^{(\lambda)}$ given as in (54), we obtain another periodic function $\tau_{d,e}^{(\lambda)}$ just by substitution in (49). Then there exists a unique (up to isometries) curve $\gamma_{d,e}^{(\lambda)}$ in the unit sphere having them as curvature and torsion functions, respectively. Therefore, we have proved the following proposition.

Proposition 9. *For any $0 < \lambda < G$, the collection $\{\gamma_{d,e}^{(\lambda)} : (e, d) \in \mathcal{D}\}$ is a two-parameter family of curves in $\mathbb{S}^3(G)$, whose curvature and torsion (given respectively in (54) and (49)) are periodic solutions of the Euler–Lagrange equations corresponding to the energy functional $\mathcal{F}^{(\lambda)}$ (40) and (41).*

There is a more natural way of writing the last proposition. Given any pair of positive real numbers (β, α), $\alpha > \beta > 0$, one may see that the parameters (e, d) defined by

$$e^2 = \frac{(G-\lambda)\alpha\beta}{(\lambda+\alpha)(\lambda+\beta)}, \qquad d = (e^2+G) - \frac{e^2\lambda^2}{\alpha\beta}, \tag{56}$$

verify (52) and (53). Combination of this with (55) allow us to express everything in terms of (β, α). Using the same symbol for the region $\mathcal{D}$ expressed with respect to the new parameters (β, α), we have $\mathcal{D} = \{(\beta, \alpha) : \alpha > \beta > 0\}$ and the following corollary.

Corollary 10. *For any $0 < \lambda < G$, there exists a two-parameter family $\mathcal{R}^{(\lambda)}_{\beta,\alpha} = \{\gamma^{(\lambda)}_{\beta,\alpha} : \alpha > \beta > 0\}$ of curves in $\mathbb{S}^3(G)$ whose curvature and torsion functions, given respectively by*

$$(\kappa^{(\lambda)}_{\beta,\alpha}(s))^2 = \frac{2(\alpha+\lambda)(\beta+\lambda)}{(\alpha+\beta+2\lambda)-(\alpha-\beta)\sin(2\sqrt{G-\lambda}s-\pi/2)} - \lambda, \tag{57}$$

$$\tau^{(\lambda)}_{\beta,\alpha} = \sqrt{\frac{(G-\lambda)\alpha\beta}{(\lambda+\alpha)(\lambda+\beta)}}\left(1+\frac{\lambda}{(\kappa^{(\lambda)}_{\beta,\alpha}(s))^2}\right) \tag{58}$$

are periodic solutions of the Euler–Lagrange equations (40) and (41) for the energy functional $\mathcal{F}^{(\lambda)}$. Moreover $\sqrt{\beta}$ and $\sqrt{\alpha}$ are the minima and the maxima of the curvature, respectively.

Members of $\mathcal{R}^{(\lambda)}_{\beta,\alpha}$ are candidates to be closed critical points of $\mathcal{F}^{(\lambda)}$. Without loss of generality we may assume $G = 1$. Then by using Proposition 6 with $P(\kappa) = (\kappa^2+\lambda)^{1/2}$, we have the following proposition.

Proposition 11. *Let $\gamma^{(\lambda)}_{\beta,\alpha}$ be a curve in $\mathbb{S}^3(1)$ belonging to $\mathcal{R}^{(\lambda)}_{\beta,\alpha}$ and $\rho(\beta,\alpha)$ the period of its curvature $\kappa^{(\lambda)}_{\beta,\alpha}(s)$. Then $\gamma^{(\lambda)}_{\beta,\alpha}$ is a closed critical point of $\mathcal{F}^{(\lambda)}$, if and only if, the angular progression of $\theta(s)$ and $\varphi(s)$ in one period of the curvature*

$$\Delta\varphi(\gamma^{(\lambda)}_{\beta,\alpha}) = \int_0^{\rho(\beta,\alpha)} \left(\frac{\lambda a}{1-a^2}\right)\frac{(\kappa^2+\lambda)^{1/2}}{\kappa^2-\lambda a^2/(1-a^2)}\,\mathrm{d}s, \tag{59}$$

$$\Delta\theta(\gamma^{(\lambda)}_{\beta,\alpha}) = \int_0^{\rho(\beta,\alpha)} \left(\frac{\lambda b}{1-b^2}\right)\frac{\left(\kappa^2+\lambda\right)^{1/2}}{\kappa^2-\lambda b^2/(1-b^2)}\,\mathrm{d}s \tag{60}$$

are rationally related to 2π (here a and b are given in (34) and to simplify the notation, we are using κ instead of $\kappa^{(\lambda)}_{\beta,\alpha}$ in the above formulas).

If we want to pick the closed critical curves out of $\mathcal{R}^{(\lambda)}_{\beta,\alpha}$, we must check the closure conditions given in Proposition 11. For any $(\beta,\alpha) \in \mathcal{D}$, let $\kappa^{(\lambda)}_{\beta,\alpha}(s)$ be the corresponding non-constant periodic solutions of (40), (41) given in (57), (58), they determine a curve $\gamma^{(\lambda)}_{\beta,\alpha}$ in $\mathbb{S}^3(1)$ belonging to $\mathcal{R}^{(\lambda)}_{\beta,\alpha}$. Then we define the map $\Lambda : \mathcal{D} \to \mathbb{R}^2$

$$\Lambda(\beta,\alpha) = \left(\frac{\Lambda\varphi(\gamma^{(\lambda)}_{\beta,\alpha})}{2\pi}, \frac{\Lambda\theta(\gamma^{(\lambda)}_{\beta,\alpha})}{2\pi}\right).$$

We remind that a and b are defined in terms of the integration constants (e, d) in (34) (and so in terms of β and α, (55)), then one may use this and (52), (53) to see that $1 > a > b > 0$

and that Λ is differentiable. To better describe the behavior of $\Lambda\theta$ and $\Lambda\varphi$ we shall introduce two new parameters ($r = r(\beta, \alpha)$, $w = (\beta, \alpha)$) that will simplify the expressions of $\Lambda\theta$ and $\Lambda\varphi$. We define (w, r) by

$$w = \frac{\lambda b^2}{1 - b^2}, \qquad r = \frac{\lambda a^2}{1 - a^2}$$

with a and b as given in (34). In terms of (w, r), $\mathcal{D}$ is defined by

$$\mathcal{D} = \left\{ (w, r) \in \frac{\mathbb{R}^2}{w} \in (0, 1 - \lambda), r > r_h(w) \right\}, \tag{61}$$

where

$$r_h(w) = \frac{(\sqrt{w} + \sqrt{1 - \lambda}(\lambda + w))^2}{(1 - \lambda - w)^2}. \tag{62}$$

By using (34), one can see that the integration constants (e, d) may be rewritten as

$$d = \frac{w}{w + \lambda} + \frac{r}{r + \lambda}, \qquad e^2 = \frac{wr}{(w + \lambda)(r + \lambda)}, \tag{63}$$

and the polynomial

$$Q(u) = (d - 1 - e^2)(u - \alpha)(u - \beta), \tag{64}$$

defined in (50) becomes

$$Q(u) = -\frac{\lambda^2}{(w + \lambda)(r + \lambda)}(u^2 - u(r + w - (\lambda + r)(\lambda + w)) + rw) \tag{65}$$

with $(w, r) \in \mathcal{D}$. We notice here some interesting relations obtained from (64) and (65) that will be useful later

$$\begin{gathered} Q(w) = -\lambda^2 w, \qquad \alpha\beta = wr, \qquad Q(r) = -\lambda^2 r, \\ \alpha + \beta = w + r - (w + \lambda)(r + \lambda), \qquad (\alpha + \lambda)(\beta + \lambda) = (1 - \lambda)(r + \lambda)(w + \lambda). \end{gathered} \tag{66}$$

Since $\kappa_{\beta,\alpha}^{(\lambda)}$ is the periodic curvature of $\gamma_{\beta,\alpha}^{(\lambda)}$, the function $u_{\beta,\alpha}^{(\lambda)} = (\kappa_{\beta,\alpha}^{(\lambda)})^2$ has period $(2\pi/\sqrt{1 - \lambda})$, increases monotonically between its minimum $u_{\beta,\alpha}^{(\lambda)}(0) = \beta$ and its maximum $u_{\beta,\alpha}^{(\lambda)}(\pi/\sqrt{1 - \lambda}) = \alpha$, and it is even about $s = (\pi/\sqrt{1 - \lambda})$. Hence, combining (51), (63) and Proposition 11, we have

$$\Lambda\theta(\gamma_{\beta,\alpha}^{(\lambda)}) = 2\int_\beta^\alpha \frac{\mathrm{d}\theta}{\mathrm{d}u}\mathrm{d}u = (w + \lambda)\sqrt{w(r + \lambda)} \int_\beta^\alpha \frac{\mathrm{d}u}{(u - w)\sqrt{(u + \lambda)(\alpha - u)(u - \beta)}},$$

and

$$\Lambda\varphi(\gamma_{\beta,\alpha}^{(\lambda)}) = 2\int_\beta^\alpha \frac{\mathrm{d}\varphi}{\mathrm{d}u}\mathrm{d}u = (r + \lambda)\sqrt{r(w + \lambda)} \int_\beta^\alpha \frac{\mathrm{d}u}{(u - r)\sqrt{(u + \lambda)(\alpha - \upsilon)(u - \beta)}}.$$

Now, using 3.137.6 of [15], one gets

$$\Lambda\theta(\gamma^{(\lambda)}_{\beta,\alpha}) = 2\frac{(\lambda+w)w^{1/2}(\lambda+r)^{1/2}}{(\alpha-w)(\lambda+\alpha)^{1/2}}\mathbf{\Pi}\left(\frac{\pi}{2}, \frac{\alpha-\beta}{\alpha-w}, \sqrt{\frac{\alpha-\beta}{\alpha+\lambda}}\right), \tag{67}$$

and

$$\Lambda\varphi(\gamma^{(\lambda)}_{\beta,\alpha}) = 2\frac{(\lambda+r)r^{1/2}(\lambda+w)^{1/2}}{(\alpha-r)(\lambda+\alpha)^{1/2}}\mathbf{\Pi}\left(\frac{\pi}{2}, \frac{\alpha-\beta}{\alpha-r}, \sqrt{\frac{\alpha-\beta}{\alpha+\lambda}}\right) \tag{68}$$

with $\mathbf{\Pi}(\pi/2, \upsilon, p)$ standing for the *Complete Elliptic Integral of Third Kind* of modulus $p = \sqrt{(\alpha-\beta)/(\alpha+\lambda)}$.

We express these two integrals in terms of the Heuman's Lambda function $\mathbf{\Lambda}_0$. From (64) and (66) we see that $r > \alpha > \beta > w > 0$, so

$$\frac{\alpha-\beta}{\alpha-r} < 0 \quad \text{and} \quad p^2 \leq \frac{\alpha-\beta}{\alpha-w} \leq 1,$$

conditions that allow us to write the *Complete Elliptic Integral of Third Kind* in terms of the *Heuman's Lambda* function [9]. Then, straightforward substitution into (67), (68) and some simplification lead to the following expressions for $\Lambda\theta(\gamma^{(\lambda)}_{\beta,\alpha})$, $\Lambda\varphi(\gamma^{(\lambda)}_{\beta,\alpha})$, which we also denote by $\Lambda\theta(\beta,\alpha)$ and $\Lambda\varphi(\beta,\alpha)$, respectively

$$\Lambda\theta(\beta,\alpha) = \Lambda\theta(\gamma^{(\lambda)}_{\beta,\alpha}) = \pi\mathbf{\Lambda}_0(\xi, p),$$

and

$$\Lambda\varphi(\beta,\alpha) = \Lambda\varphi(\gamma^{(\lambda)}_{\beta,\alpha}) = -2r^{1/2}\frac{(\lambda+w)^{1/2}}{(\lambda+\alpha)^{1/2}}\mathbf{K}(p) - \pi\mathbf{\Lambda}_0(\upsilon, p),$$

where $\xi = \arcsin((\lambda+w)^{1/2})/(\lambda+\beta)^{1/2})$, $\upsilon = \arcsin((\lambda+\alpha)^{1/2})/((\lambda+r)^{1/2})$, $\mathbf{\Lambda}_0(\upsilon, p)$ is the Heuman's Lambda function of modulus $p = \sqrt{(\alpha-\beta)/(\alpha+\lambda)}$ and argument υ, and $\mathbf{K}(p)$ is the *Complete Elliptic Integral of First Kind* of modulus p.

Let us denote by $\mathcal{L}_1 \cup \mathcal{L}_2$ the frontier of $\mathcal{D}$, where $\mathcal{L}_1 = \{(\beta,\alpha)/\beta = 0\}$ and $\mathcal{L}_2 = \{(\beta,\alpha)/\beta = \alpha\}$ (they can be written respectively in terms of (w, r) as $\{(0,r)/r \geq (\lambda^2/(1-\lambda))\}$ and $\{(w, r_h(w))/w \geq 0\}$, $r_h(w)$ given in (62)). We continuously extend Λ to $\mathcal{L}_1 \cup \mathcal{L}_2$. Taking limits, one obtains

$$\Lambda\varphi(0,\alpha) = \lim_{\beta\to 0}\Lambda\varphi(\beta,\alpha) = \frac{-2\sqrt{\lambda r}}{\sqrt{(1-\lambda)(\lambda+r)}}\mathbf{K}(p) - \pi\mathbf{\Lambda}_0(\arcsin\sqrt{1-\lambda}, p).$$

Thus we have

$$\lim_{p\to 0}\Lambda\varphi(0,\alpha) = \lim_{\alpha\to 0}\Lambda\varphi(0,\alpha) = \frac{-\pi}{\sqrt{1-\lambda}},$$

and

$$\lim_{p\to 1}\Lambda\varphi(0,\alpha) = \lim_{\alpha\to +\infty}\Lambda\varphi(0,\alpha) = -\infty.$$

Actually, numeric computations show that for any $\lambda \in (0, 1)$, $\Lambda\varphi(0, \alpha)$ decreases monotonically from $(-\pi/\sqrt{1-\lambda})$ to $-\infty$ as α increases from 0 to $+\infty$. Moreover $\Lambda\theta(0, \alpha) = \lim_{\beta\to 0} \Lambda\theta(\beta, \alpha) = \pi$ and

$$\Lambda\varphi(\alpha, \alpha) = \lim_{\beta\to\alpha} \Lambda\varphi(\beta, \alpha) = -\pi\left((1-\lambda)\left(1-\left(\frac{w}{1-\lambda}\right)^{1/2}\right)\right)^{-(1/2)},$$

$$\Lambda\theta(\alpha, \alpha) = \lim_{\beta\to\alpha} \Lambda\theta(\beta, \alpha) = \pi\left(1-\left(\frac{w}{1-\lambda}\right)^{1/2}\right)^{1/2}$$

with $w \in [0, 1-\lambda)$. Therefore, we have seen that

$$\Lambda(\mathcal{L}_1) = \left\{\left(x, \frac{1}{2}\right); x \le -\frac{1}{2\sqrt{1-\lambda}}\right\},$$

and

$$\Lambda(\mathcal{L}_2) = \left\{(x, y) \in \mathbb{R}^2; 0 < y \le \frac{1}{2} \text{ and } xy = -\frac{1}{4\sqrt{1-\lambda}}\right\}.$$

Finally, observe that the curve defined by $d = e^2 + G$ in the first description of $\mathcal{D}$ (52) and (53), corresponds to the situation in which the curvature α grows toward $+\infty$. In this case, from the relations between r and α that can be obtained from (66), one gets $\lim_{\alpha\to+\infty} \Lambda\varphi(\beta, \alpha) = -\infty$ and $\lim_{\alpha\to+\infty} \Lambda\theta(\beta, \alpha) = 2\arcsin((1 - w/(1-\lambda))^{1/2})$, that varies from π to 0 as w moves from 0 to $1-\lambda$, and so $\Lambda\theta/2\pi$ varies from (1/2) to 0. Hence, we obtain (see Fig. 2)

$$\Lambda(\mathcal{D}) = \left\{(x, y); x < 0 \text{ and } \frac{1}{2} > y > -\frac{1}{4x\sqrt{1-\lambda}}\right\}. \quad (69)$$

Non-constant solutions of the Euler–Lagrange equations for the functional $\mathcal{F}^{(\lambda)}$ can be indexed in the region $\Lambda(\mathcal{D})$ and they are periodic functions. Hence, critical curves for the different boundary problems included in $\mathcal{H}$, are indexed in subsets of $\Lambda(\mathcal{D})$. By Proposition 11,

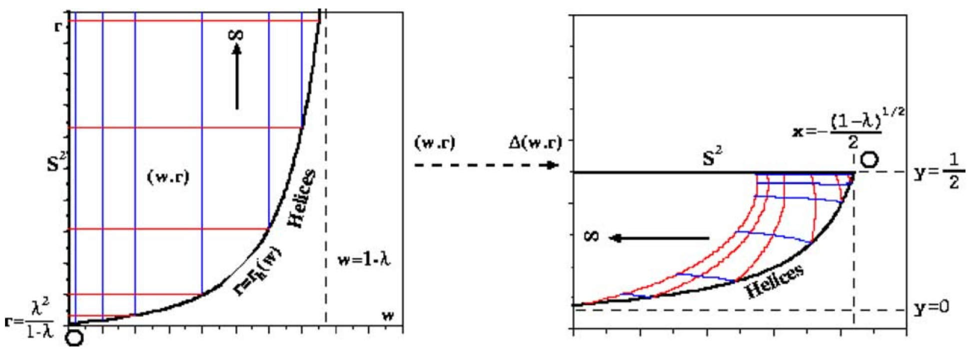

Fig. 2. $\Lambda(\mathcal{D}) = \{(x, y) : x < 0, 1/2 > y > -1/(4x\sqrt{1-\lambda})\}$.

a curve $\gamma^{(\lambda)}_{\alpha,\beta} \in \mathcal{R}^{(\lambda)}_{\alpha,\beta}$ will be a closed critical point of $\mathcal{F}^{(\lambda)}$, if and only if, $\Lambda(\alpha, \beta) \in \mathbb{Q}^2$ and we see from (69) that $\Lambda(\mathcal{D}) \cap \mathbb{Q}^2 \neq \emptyset$ and then the family of closed critical points of $\mathcal{F}^{(\lambda)}$ is indexed on it. This determines completely the space of closed critical points of $\mathcal{F}^{(\lambda)}$ in $\mathbb{S}^3(1)$. It can be parameterized (modulo multiple covers of the curves and isometries of the sphere) in the plane region $\overline{\Lambda(\mathcal{D})} \cap \mathbb{Q}^2$. Points in the upper border $\Lambda(\mathcal{L}_1) \cap \mathbb{Q}^2$ represent critical points of $\mathcal{F}^{(\lambda)}$ that lie in $\mathbb{S}^2(1)(\beta = 0 \Leftrightarrow e = 0 \Leftrightarrow \tau = 0)$ and have been studied in [4], while points in the lower border $\Lambda(\mathcal{L}_2) \cap \mathbb{Q}^2$ correspond to closed critical helices fully immersed in $\mathbb{S}^3(1)$ $(\alpha = \beta \Leftrightarrow d = \lambda + 2e\sqrt{G - \lambda})$ and we described them in Proposition 7. Observe that the intersection of the two borders, the "vertex" $(-1/(2\sqrt{1-\lambda}), 1/2)$, is associated to geodesics of $\mathbb{S}^3(1)$. Points inside the region $\Lambda(\mathcal{D}) \cap \mathbb{Q}^2$ correspond to non-constant curvature closed critical curves fully immersed in $\mathbb{S}^3(1)$. This describes all the closed critical curves of $\mathcal{F}^{(\lambda)}$ in $\mathbb{S}^3(1)$ (multiple covers of a closed elastica correspond to the same point of $\overline{\Lambda(\mathcal{D})} \cap \mathbb{Q}^2$). Then, we have in particular.

Proposition 12. *For any* $\lambda \in (0, 1)$ *and for any choice of natural parameters* $n, m, l \in \mathbb{N}$ *satisfying* $(n, m, l) = 1$, $0 < l < n/2$ *and* $n^2 < 4ml\sqrt{1-\lambda}$, *there exists a closed critical point* $\gamma^{(\lambda)}_{n,m,l}$ *of* $\mathcal{F}^{(\lambda)}$ *in* $\mathbb{S}^3(1)$ *that closes up after* n *periods of its curvature,* m *trips around the* "equator" *of* x_φ *and* l *trips around the* "equator" *of* x_θ. *For any choice of natural numbers verifying the above conditions, a closed critical point* $\gamma^{(\lambda)}_{m,n,l}$ *is totally determined in* (88), (89), (57) and (58). *Every closed critical point of* $\mathcal{F}^{(\lambda)}$ *in* $\mathbb{S}^3(1)$ *can be obtained in this way*.

Fig. 3 shows the stereographic projection from $\mathbb{S}^3(1)$ of closed non-constant curvature $\mathcal{F}^{(\lambda)}$-critical curves $\gamma^{(\lambda)}_{n,m,l}$, *numerically obtained for certain values of* λ, n, m and l.

Fig. 3. Stereographic projections of the closed $\mathcal{F}^{(\lambda)}$-critical curves $\gamma^{(0.5)}_{36,47,7}$ and $\gamma^{(0.3)}_{45,40,7}$.

5. Closed elastic curves

Langer and Singer discovered that there is a cylindrical coordinate system naturally associated to an elastic curve ($P(\kappa) = \kappa^2 + \lambda$) in $\mathbb{R}^3$ [21]. They used it to explicitly integrate the Frenet equations of an elastica and to describe the closed (non-free) elastic curves in Euclidean three-space. In particular there are no closed free elastic curves in $\mathbb{R}^3$. On the other hand, the cylindrical coordinate system (29) was used in [22] to show that the elastic curves coordinates ($P(\kappa) = \kappa^2 + \lambda$) in $\mathbb{S}^3(G)$ might be obtained with the aid of elliptic functions. However, they did not explicitly integrate the equations nor considered the problem of closed critical elasticae in the three-sphere. We shall deal with this issues in this section. We consider the elastic functional acting on a suitable space $\mathcal{H}$ of immersed curves in $\mathbb{S}^3(G)$

$$\mathcal{F}(\gamma) = \int_\gamma \kappa^2 \,\mathrm{d}s, \tag{70}$$

where $\kappa(s)$ is the curvature of $\gamma(s)$. A curve $\gamma \in \mathcal{H}$ is said to be an (*free*) *elastic curve* (or simply *elastica*) if it is a critical point of $\mathcal{F}$. Therefore it satisfies the Euler–Lagrange equations (5) and (6) with $P(\kappa) = \kappa^2$

$$2\kappa_{ss} + \kappa^3 + 2\kappa(G - \tau^2) = 0, \tag{71}$$

$$4\frac{\mathrm{d}}{\mathrm{d}s}(\tau\kappa^2) = 0. \tag{72}$$

5.1. *Closed critical elastica of constant curvature*

We first investigate the existence of closed critical curves γ of constant curvature. By using a similar argument to that of Section 4.1 one can determine the closed elasticae constant curvature (take $r = 2$ in Proposition 16). Since we are considering curves fully immersed in $\mathbb{S}^3(1)$, geodesics, which are global minima of $\mathcal{F}$, do not appear there. Fig. 4 shows the stereographic projection of the elastic helices corresponding to $q = 1$ and 1/32.

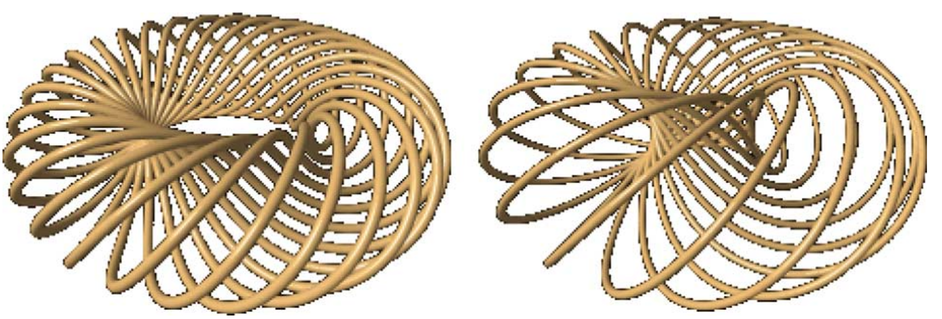

Fig. 4. Stereographic projections of the closed elastic helices γ_1 and $\gamma_{1/32}$.

5.2. Closed critical elasticae of non-constant curvature

Assume now that κ is a non-constant function. By a direct computation or by applying the more general Proposition 4, we get the first integrals of the *Euler–Lagrange* equations (21) and (22)

$$16\kappa^2\kappa_s^2(s) = 4d\kappa^2 - 16G\kappa^4 - 4\kappa^6 - e^2, \tag{73}$$

$$\tau(s) = \left(\frac{e}{4\kappa^2(s)}\right), \tag{74}$$

where d and e are the constants of integration. Writing $u(s) = \kappa^2(s)$ and

$$Q(x) = -(x^3 + 4Gx^2 - \mathrm{d}x + \tfrac{1}{4}e^2). \tag{75}$$

Eq. (73) becomes

$$u_s^2 = Q(u) = -u^3 - 4Gu^2 + \mathrm{d}u - \tfrac{1}{4}e^2. \tag{76}$$

In order to find closed critical curves, we need this equation to have periodic solutions. Then $Q(u)$ must have two positive roots $\alpha > \beta > 0$ and a negative one, α_0, such that $Q(u) = (\alpha - u)(u - \beta)(u - \alpha_0)$, where constants of integration and roots are related by

$$\alpha_0 = -(4G + \alpha + \beta), \tag{77}$$

$$e^2 = 4(4G + \alpha + \beta)\alpha\beta, \tag{78}$$

$$d = (\alpha + \beta)(4G + \alpha + \beta) - \alpha\beta. \tag{79}$$

Conversely, for any couple (β, α) with $\alpha > \beta > 0$, we can use the above relations to obtain constants $d > 0$ and $e > 0$ in such a way that α and β are positive roots of the polynomial $Q(u)$ defined in (76). So if we look for the solution $u(s)$ of (76) with initial condition $u(0) = \beta$, then using for example formula 3.131:6 of [15], we see that it is a periodic function $u_{d,e}(s) = u_{\beta,\alpha}(s)$ given by

$$u_{\beta,\alpha}(s) = \kappa_{\beta,\alpha}^2(s) = \alpha - (\alpha - \beta)\mathrm{sn}^2(\tfrac{1}{2}(\sqrt{\alpha - \alpha_0})s - \mathbf{K}(p), p) \tag{80}$$

with $\mathbf{K}(p)$ standing for the Complete Elliptic Integral of the First Kind of modulus $p = \sqrt{(\alpha - \beta)/(\alpha - \alpha_0)}$. The minima and the maxima of the above solutions are reached at $u_{\beta,\alpha}(0) = \beta$ and $u_{\beta,\alpha}(2/(\sqrt{\alpha - \alpha_0})\mathbf{K}(p)) = \alpha$, respectively. Therefore, we have proved the following proposition.

Proposition 13. *There exists a two-parameter family* $\mathcal{R}_{\beta,\alpha} = \{\gamma_{\beta,\alpha} : \alpha > \beta > 0\}$ it of curves in $\mathbb{S}^3(G)$ *whose curvature and torsion functions* $\kappa_{\beta,\alpha}$ *and* $\tau_{\beta,\alpha}$, *given respectively in* (80) *and* (74), *are periodic solutions of the Euler–Lagrange equations* (71) *and* (72) *for the elastic energy functional* $\mathcal{F}$. *Moreover* $\sqrt{\beta}$ *and* $\sqrt{\alpha}$ *are the minima and the maxima of the curvature, respectively*.

Members of $\mathcal{R}_{\beta,\alpha}$ are candidates to be closed critical points of $\mathcal{F}$. Without loss of generality we may assume $G = 1$. Then by using Proposition 6 with $P(\kappa) = \kappa^2$, we have the following proposition.

Proposition 14. *Let a $\gamma_{\beta,\alpha}$ be a curve in $\mathbb{S}^3(1)$ belonging to $\mathcal{R}_{\beta,\alpha}$ and $\rho(\beta,\alpha)$ the period of its curvature $\kappa_{\beta,\alpha}(s)$. Then $\gamma_{\beta,\alpha}$ is a closed critical point of $\mathcal{F}$ if and only if, the angular progressions of $\theta(s)$ and $\varphi(s)$ in one period of the curvature*

$$\Delta\theta(\gamma_{\beta,\alpha}) = -\frac{b}{4}\int_0^{\rho(\beta,\alpha)} \left(\frac{\kappa^2}{\kappa^2-(b^2/4)}\right) \mathrm{d}s, \tag{81}$$

$$\Delta\varphi(\gamma_{\beta,\alpha}) = -\frac{a}{4}\int_0^{\rho(\beta,\alpha)} \left(\frac{\kappa^2}{\kappa^2-(a^2/4)}\right) \mathrm{d}s \tag{82}$$

are rationally related to 2π (here a and b are in (34) and to simplify the notation, we are using κ instead of $\kappa_{\beta,\alpha}$ in the above formulas).

If we want to determine the closed critical curves of $\mathcal{R}_{\beta,\alpha}$, we must check the closure conditions given in Proposition 14. Take the region $\mathcal{D} = \{(\beta,\alpha); \alpha > \beta > 0\}$. For any $(\beta,\alpha) \in \mathcal{D}$, let $\kappa_{\beta,\alpha}(s)$ be the corresponding non-constant periodic solutions of (71), it determines a curve $\gamma_{\beta,\alpha}$ in $\mathbb{S}^3(1)$ belonging to $\mathcal{R}_{\beta,\alpha}$. Since we can exchange (β,α) and (e,d) by means of (78), (79), we indistinctly denote this curve by $\gamma_{e,d}$ or $\gamma_{\beta,\alpha}$. Then we define the map $\Lambda : \mathcal{D} \to \mathbb{R}^2$

$$\Lambda(\beta,\alpha) = \left(\frac{\Lambda\varphi(\gamma_{\beta,\alpha})}{2\pi}, \frac{\Lambda\theta(\gamma_{\beta,\alpha})}{2\pi}\right).$$

Again, we have that a and b are related to the constants of integration (e,d) in (34) (and so they are to β and α (78), (79)). Thus we have from (77), (78), (79) that $b^2/4 < \beta, \alpha < a^2/4$ and that $\Lambda : \mathcal{D} \to \mathbb{R}^2$ is differentiable.

We define new parameters (w, r) by

$$w = \tfrac{1}{4}b^2 \quad \text{and} \quad r = \tfrac{1}{4}a^2.$$

Then, by using (34), one can see that the integration constants (e,d) may be rewritten as

$$d = 4(w+r), \qquad e^2 = 16wr, \tag{83}$$

and the polynomial defined in (75) becomes

$$Q(u) = -(u^3 + 4(u-r)(u-w)) \tag{84}$$

with $r > \alpha > \beta > w > 0$. We note now some interesting relations obtained from (75) and (84) that will be useful later

$$Q(w) = -w^3 < 0, \tag{85}$$

$$Q(r) = -r^3 < 0, \tag{86}$$

and

$$4wr = -\alpha_0\alpha\beta, \qquad 4(w+r) = -(\alpha\beta + \alpha_0(\alpha+\beta)). \tag{87}$$

The function $u_{\beta,\alpha} = \kappa^2_{\beta,\alpha}(s)$ has period $4/(\sqrt{\alpha-\alpha_0})\mathbf{K}(p)$, increases monotonically between its minimum $u_{\beta,\alpha}(0) = \beta$ and its maximum $u_{\beta,\alpha}(2/(\sqrt{\alpha-\alpha_0})\mathbf{K}(p)) = \alpha$, and it is even about $s = 2/(\sqrt{\alpha-\alpha_0})\mathbf{K}(p)$. Hence, using (76) and (83) in (81) and (82), we have

$$\Lambda\theta(\gamma_{\beta,\alpha}) = 2\int_\beta^\alpha \frac{\mathrm{d}\theta}{\mathrm{d}u}\,\mathrm{d}u = -w^{1/2}\int_\beta^\alpha \frac{u\,\mathrm{d}u}{(u-w)\sqrt{(\alpha-u)(u-\beta)(u-\alpha_0)}},$$

and

$$\Lambda\varphi(\gamma_{\beta,\alpha}) = 2\int_\beta^\alpha \frac{\mathrm{d}\varphi}{\mathrm{d}u}\,\mathrm{d}u = -r^{1/2}\int_\beta^\alpha \frac{u\,\mathrm{d}u}{(u-r)\sqrt{(\alpha-u)(u-\beta)(u-\alpha_0)}}.$$

Thus, using 3.137.6 of [15], one gets

$$\Lambda\theta(\gamma_{\beta,\alpha}) = -2\left(\frac{w}{\alpha-\alpha_0}\right)^{1/2}\left(\mathbf{K}(p) + \frac{w}{\alpha-w}\mathbf{\Pi}\left(\frac{\pi}{2}, \frac{\alpha-\beta}{\alpha-w}, \sqrt{\frac{\alpha-\beta}{\alpha-\alpha_0}}\right)\right), \tag{88}$$

and

$$\Lambda\varphi(\gamma_{\beta,\alpha}) = -2\left(\frac{r}{\alpha-\alpha_0}\right)^{1/2}\left(\mathbf{K}(p) + \frac{r}{\alpha-r}\mathbf{\Pi}\left(\frac{\pi}{2}, \frac{\alpha-\beta}{\alpha-r}, \sqrt{\frac{\alpha-\beta}{\alpha-\alpha_0}}\right)\right) \tag{89}$$

with $\mathbf{\Pi}(\pi/2, \upsilon, p)$ standing for the *Complete Elliptic Integral of Third Kind* of modulus $p = \sqrt{(\alpha-\beta)/(\alpha-\alpha_0)}$ and $\mathbf{K}(p)$ represents the *Complete Elliptic Integral of the First Kind* of modulus p.

Now we express these two integrals in terms of the Heuman's Lambda function Λ_0. From (84) and (85) we see that $r > \alpha > \beta > w > 0$, so

$$\frac{\alpha-\beta}{\alpha-r} < 0 \quad \text{and} \quad p^2 \le \frac{\alpha-\beta}{\alpha-w} \le 1,$$

conditions that allow us writing the *Complete Elliptic Integral of Third Kind* in terms of the *Heuman's Lambda* function [9]. Then, straightforward substitution and simplification into (88), (89), lead to the following expressions for $\Lambda\theta(\gamma_{\beta,\alpha})$, $\Lambda\varphi(\gamma_{\beta,\alpha})$, which we also denote by $\Lambda\theta(\beta,\alpha)$ and $\Lambda\varphi(\beta,\alpha)$, respectively

$$\Lambda\theta(\gamma_{\beta,\alpha}) = -\frac{2w^{1/2}}{(\alpha-\alpha_0)^{1/2}}\mathbf{K}(p) - \pi\mathbf{\Lambda}_0(\xi, p), \tag{90}$$

and

$$\Lambda\varphi(\gamma_{\beta,\alpha}) = -\frac{2r^{1/2}}{(\alpha-\alpha_0)^{1/2}}\left(1 - \frac{r}{r-\beta}\right)\mathbf{K}(p) - \pi(\mathbf{\Lambda}_0(\upsilon, p) - 1), \tag{91}$$

where $\xi = \arcsin((w-\alpha_0)/(\beta-\alpha_0))^{1/2}$ and $\upsilon = \arcsin((r-\alpha)/(r-\beta))^{1/2}$.

We continuously extend Λ to the frontier of $\mathcal{D}$. Let $\mathcal{L}_1 = \{(0,\alpha)/\alpha \ge 0\}$ and $\mathcal{L}_2 = \{(\alpha,\alpha)/\alpha \ge 0\}$ be the frontier curves. Taking limits, one obtains

$$\Lambda\varphi(0,\alpha) = \lim_{\beta\to 0}\Lambda\varphi(\beta,\alpha) = -\pi\left[\mathbf{\Lambda}_0\left(\arcsin\sqrt{\frac{r-\alpha}{r}}, p\right) - 1\right],$$

that decreases monotonically from π to 0, as α increases from 0 to $+\infty$, and

$$\Lambda\theta(0,\alpha) = \lim_{\beta\to 0} \Lambda\theta(\beta,\alpha) = -\pi.$$

Thus, we have seen that

$$\Lambda(\mathcal{L}_1) = \left\{\left(x, -\frac{1}{2}\right); 0 \le x \le \frac{1}{2}\right\}.$$

Moreover,

$$\Lambda\varphi(\alpha,\alpha) = \lim_{\beta\to\alpha} \Lambda\varphi(\beta,\alpha) = \frac{r^{1/2}}{2(r-\alpha)}\frac{\alpha}{(\alpha-\alpha_0)^{1/2}}\pi, \quad (92)$$

$$\Lambda\theta(\alpha,\alpha) = \lim_{\beta\to\alpha} \Lambda\theta(\beta,\alpha) = \frac{w^{1/2}}{2(w-\alpha)}\frac{\alpha}{(\alpha-\alpha_0)^{1/2}}\pi. \quad (93)$$

Therefore, combining (92), (93), (84), (85) and (87) with $\alpha = \beta$, we get

$$\Lambda(\mathcal{L}_2) = \left\{(x,y) \in \mathbb{R}^2; -\frac{1}{\sqrt{2}} \le y \le -\frac{1}{2}, x > 0 \text{ and } x^2 + y^2 = \frac{1}{2}\right\}.$$

Finally, by using the relations between r and α that can be obtained from (86), (87) and (90), (91), one gets $\lim_{\alpha\to+\infty}\Lambda\theta(\beta,\alpha) = -\pi$ and $\lim_{\alpha\to+\infty}\Lambda\varphi(\beta,\alpha) = 0$, so

$$\lim_{\alpha\to+\infty}\Lambda(\beta,\alpha) = (0, -\tfrac{1}{2}).$$

Then, we have that (see Fig. 5)

$$\Lambda(\mathcal{D}) = \left\{(x,y); x^2 + y^2 < \frac{1}{2}, x > 0 \text{ and } y < -\frac{1}{2}\right\}.$$

Non-constant solutions of the Euler–Lagrange equations for the elastic functional $\mathcal{F}$ in $\mathbb{S}^3(1)$ can be indexed in the region $\Lambda(\mathcal{D})$ and they are periodic functions. Hence, elastic curves for the different boundary problems included in $\mathcal{H}$, are indexed in subsets of $\Lambda(\mathcal{D})$. By

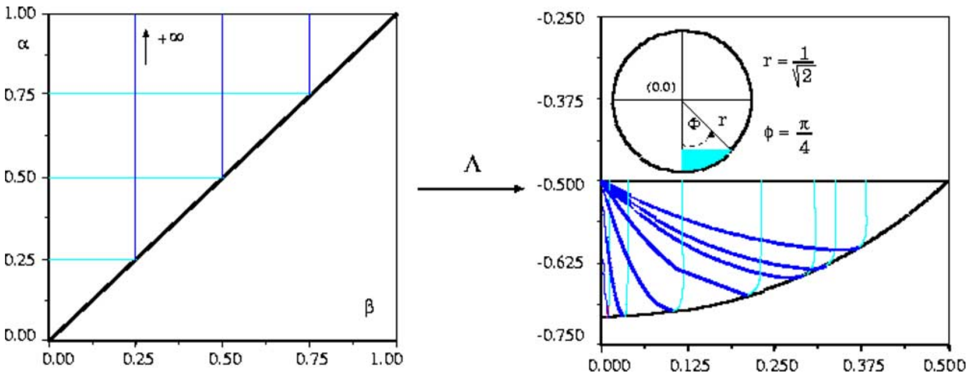

Fig. 5. $\Lambda(\mathcal{D}) = \{(x,y); x^2+y^2 < 1/2, x > 0 \text{ and } y < -(1/2)\}$.

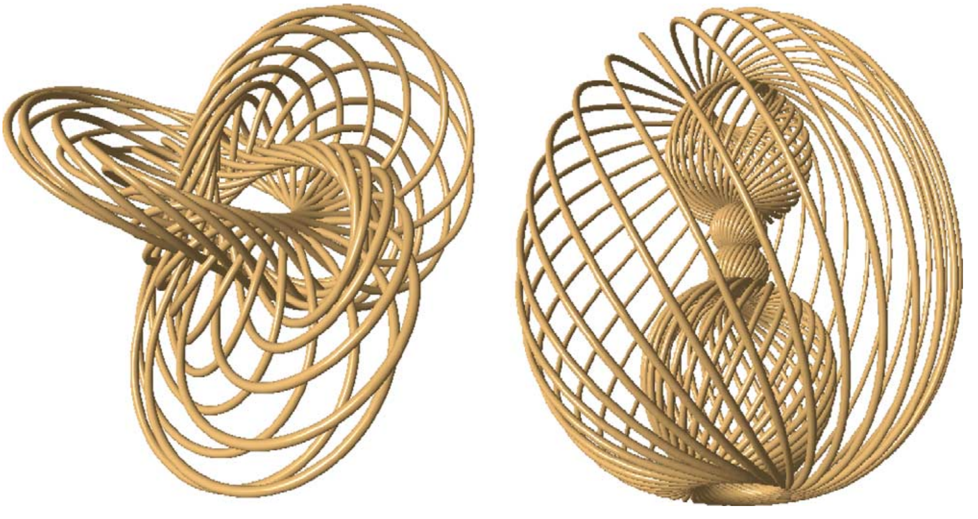

Fig. 6. Stereographic projections of the closed elasticae $\gamma_{75,22,47}$ and $\gamma_{150,30,97}$.

Proposition 14, a curve $\gamma_{\beta,\alpha} \in \mathcal{R}_{\beta,\alpha}$ will be closed if and only if $\Lambda(\beta,\alpha) \in \mathbb{Q}^2$. Hence, in the above setting (see Fig. 5), closed non-constant curvature elastic curves fully immersed in $\mathbb{S}^3(1)$ are indexed in $\Lambda(\mathcal{D}) \cap \mathbb{Q}^2$. Points in the upper boundary of this region, $\Lambda(\mathcal{L}_1) \cap \mathbb{Q}$, represent closed elastic curves that lie in $\mathbb{S}^2(1)$ [20]. Here, geodesics correspond to the "vertex" of the region, $((1/2), -(1/2))$. Points in the lower boundary, $\Lambda(\mathcal{L}_2) \cap \mathbb{Q}$, correspond to closed elastic helices fully immersed in $\mathbb{S}^3(1)$. These are all the closed elastic curves in $\mathbb{S}^3(1)$ (multiple covers of a closed elastica correspond to the same point of $\overline{\Lambda(\mathcal{D})} \cap \mathbb{Q}^2$). Then, we have in particular the following proposition.

Proposition 15. *For any choice of natural parameters $m, n, l \in \mathbb{N}$ satisfying $(n, m, l) = 1, 0 < m < n/2 < l < n/\sqrt{2}$ and $m^2 + l^2 < n^2/2$, there exists a closed elastica $\gamma_{n,m,l}$ fully immersed in $\mathbb{S}^3(1)$ that closes up after n periods of its curvature, m trips around the "equator" of x_φ and l trips around the "equator" of x_θ. For any choice of natural numbers verifying the above conditions, a closed critical point $\gamma_{n,m,l}$ is totally determined in (80) and (74). Every closed elastica in $\mathbb{S}^3(1)$ can be obtained in this way.*

Fig. 6 shows the stereographic projection from $\mathbb{S}^3(1)$ of closed non-constant curvature elastic curves $\gamma_{n,m,l}$, numerically obtained for certain values of n, m and l.

5.3. Remark

Free elastic helices given in Proposition 16 with $r = 2$, were used in [5] to construct examples of constant mean curvature Chen–Willmore tori in the anti-De Sitter space H_1^4. In a similar way, the above proposition will provide examples of non-constant mean curvature Chen–Willmore tori in H_1^4.

6. Closed hyperelastic curves

Assume now that we have *curvature energy* functionals of the type $\mathcal{F}^r(\gamma) = \int_\gamma \kappa^r \, \mathrm{d}s$, with r a natural number greater than 1, acting on closed curves of $\mathbb{S}^3(G)$. The argument to prove the existence of closed constant curvature critical curves γ of $\mathcal{F}^r$ (*free hyperelastic curves*) goes much in the line of the previous cases. The following proposition was obtained in [5].

Proposition 16. *For any natural number $r \geq 2$, the set of constant curvature closed critical curves of $\int_\gamma \kappa^r \, \mathrm{d}s$ in $\mathbb{S}^3(G)$ (and therefore, also with constant non-zero torsion: helices) forms a rational one-parameter family $\{\gamma_q : q \in \mathbb{Q}^+ - \{1/2\}\}$. Moreover, they can be determined by a similar procedure to that of Section 4.1.*

As for the non-constant curvature case, the Euler–Lagrange equations (5), (6) give

$$r(r-1)(r-2)\kappa^{r-3}\kappa_s^2 + r(r-1)\kappa^{r-2}\kappa_{ss} + r(\kappa^2+1)\kappa^{r-1} - \kappa^{r+1} = \frac{e^2}{r^3\kappa^{r-3}} \quad (94)$$

with $e \in \mathbb{R}$. Considering it as an autonomous system, the equation of orbits can be deduced from its first integrals (Proposition 4) which in this case turns out to be

$$r^2(r-1)^2\kappa^{2r-4}y^2 = d - (r-1)^2\kappa^{2r} - r^2\kappa^{2r-2} - \frac{e^2}{r^2\kappa^{2r-2}} \quad (95)$$

with $d > 0$. We denote by $q(\kappa) = dr^2\kappa^{2r-2} - r^2(r-1)^2\kappa^{4r-2} - r^4\kappa^{4r-4}$, then $q(x)$ has a positive root ε. Denote by δ the maximum of $q(x)$ between 0 and ϵ. Hence, any couple $d, e \in \mathbb{R}$, satisfying $0 < d, 0 < e^2 < \delta$, gives rise to a periodic orbit and, therefore, (94) has positive periodic solutions. By substitution of one of these periodic solutions in Proposition 6 one might see that the functions (37) and (38) moves continuously as d, e does in the specified range of variation. Therefore, they hit rational multiples of 2π and we would find closed critical points.

Proposition 17. *For any natural number $r \geq 2$, there exist closed free hyperelastic curves of non-constant curvature in $\mathbb{S}^3(G)$.*

Observe that this result is not true in $\mathbb{S}^2(G)$ [4].

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